

FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATION WITH DISCONTINUOUS DIFFUSION

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ABSTRACT. In this paper we study a stochastic differential equation driven by a fractional Brownian motion with a discontinuous coefficient. We also give an approximation to the solution of the equation. This is a first step to define a fractional version of the skew Brownian motion.

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1. Introduction

In this article we show existence and uniqueness for a stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H > 1/2$ and a discontinuous diffusion coefficient. Recall that a fractional Brownian motion B^H , is a centered Gaussian process with covariance structure given by:

$$R_H(t, s) = \mathbb{E}(B_t^H B_s^H) = \frac{1}{2} \{|t|^{2H} + |s|^{2H} - |t - s|^{2H}\}.$$

The particular case $H = 1/2$ results the Brownian motion.

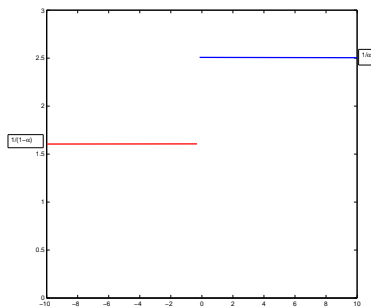
The equation we will consider in this article is given by:

$$x_t = x_0 + \int_0^t \sigma(x_s) dB_s^H, \quad t \geq 0; \tag{1.1}$$

where σ is a discontinuous function given by

$$\sigma(x) = \frac{1}{\alpha} 1\{x \geq 0\} + \frac{1}{1-\alpha} 1\{x < 0\},$$

for some $\alpha \in (0, 1)$. Note that, without loss of generality, we can assume that $\alpha < 1/2$. The following picture shows the function σ .

FIGURE 1. $\sigma(x)$ for $\alpha = 0.4$

Nakao [6] in 1972, solved the problem of pathwise uniqueness of solutions of SDE driven by Brownian motion, ($H = 1/2$), and with a diffusion coefficient uniformly positive and of bounded variation on any compact interval. The author proved the pathwise uniqueness holds for such equation. After that, many works have been developed in this direction, in order to show existence and/or uniqueness for SDE with general coefficients in the diffusion. We can refer here [7] and [3] and the references therein among others. In the case $H > 1/2$, the only cases of stochastic differential equations driven a fractional Brownian motion with discontinuous coefficients which have been studied are those corresponding to discontinuous drift coefficient (for $H > 1/2$). Regarding that, in [4], the authors studied a Hölder continuous drift except on a finite numbers of points. Other class of discontinuity in SDE driven a fractional Brownian motion is related to add to a Poisson process for the SDE. In [2], the authors proved the strong solution of this kind of SDE driven by fBm and Poisson point process extending the results given in [4].

The main interest in working in this type of equations is related with the problem to define a fractional version of the Skew Brownian motion. In the Brownian motion framework the Skew Brownian motion appeared as a natural generalization of the Brownian motion. The Skew Brownian motion is a process that behaves like a Brownian motion except that the sign of each excursion is chosen using an independent Bernoulli random variable of parameter $p \in (0, 1)$. For $p = 1/2$ the process corresponds to a Brownian motion. This process is a markov process, semi-martingale which is a strong solution to some Stochastic Differential Equation (SDE) with local time, (see [3] for a survey).

$$X_t = x + B_t + (2p - 1)L_t^0(X). \quad (1.2)$$

In the case of Brownian motion and from Itô-Tanaka formula, equation (1.1) and (1.2) are equivalent. In the context of fractional Brownian motion, since Tanaka formula only exists for fractional Brownian motion and some functionals on it, there is no relation between both equations. Is in this sense and until our knowledge this work corresponds to the first step in this direction.

We organized our paper as follows. In Section 2, we give some preliminaries related to fractional calculus. Section 3 is devoted to analyze the problem according to the initial condition and the Scheme approximation. The main results is presented in Section 4. A simulation study is reported in last Section.

2. Preliminaries

This section is devoted to introduce an extension of Young's integral defined by Zähle in [9]. Before giving the definition of this integral, we establish some notations and definitions that we use in this paper.

Consider $0 \leq a < b \leq T$, $\alpha \in (0, 1)$, $p > 1$ and $f \in L^p([0, T])$. For $t \in [a, b]$ we set

$$D_{a+}^\alpha f(t) = L^p - \lim_{\varepsilon \downarrow 0} \frac{1}{\Gamma(1-\alpha)} \left(\frac{f_t}{(t-a)^\alpha} + \alpha \int_a^{t-\varepsilon} \frac{f_t - f_r}{(t-r)^{1+\alpha}} dr \right), \quad (2.1)$$

in case that this limit is well-defined, where we use the convention $f_r = 0$ on $[a, b]^c$. In this case $D_{a+}^\alpha f$ is called the left-fractional derivative of f of order α . Similarly, for $f \in L^p([0, T])$ and $t \in [a, b]$, the right-fractional derivative of f of order α is introduced as

$$D_{b-}^\alpha f(t) = L^p - \lim_{\varepsilon \downarrow 0} \frac{1}{\Gamma(1-\alpha)} \left(\frac{f_t}{(b-t)^\alpha} + \alpha \int_{t+\varepsilon}^b \frac{f_t - f_r}{(r-t)^{1+\alpha}} dr \right). \quad (2.2)$$

It is not difficult to see that, as a consequence of the proof of [8, Theorem 13.2], the fact that f , $\frac{f(\cdot)}{(\cdot-a)^\alpha}$ and $\int_a^\cdot \frac{f(\cdot)-f_r}{(\cdot-r)^{1+\alpha}} dr$ (resp. $\frac{f(\cdot)}{(b-\cdot)^\alpha}$ and $\int^\cdot \frac{f(\cdot)-f_r}{(r-\cdot)^{1+\alpha}} dr$) belong to $L^p([a, b])$ implies that $D_{a+}^\alpha f$ (resp. $D_{b-}^\alpha f$) is well-defined and

$$D_{a+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f_t}{(t-a)^\alpha} + \alpha \int_a^t \frac{f_t - f_r}{(t-r)^{1+\alpha}} dr \right), \quad (2.3)$$

(resp.

$$D_{b-}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f_t}{(b-t)^\alpha} + \alpha \int_t^b \frac{f_t - f_r}{(r-t)^{1+\alpha}} dr \right).$$

The space of all the α -Hölder continuous functions on $[a, b]$ is denoted by $C^\alpha([a, b])$. Then if $f \in C^\alpha([a, b])$, the norm of f is defined as follows

$$\|f\|_{\alpha, [a, b]} := \|f\|_\infty + \sup_{a \leq s < t \leq b} \frac{|f(t) - f(s)|}{(t-s)^\alpha},$$

where $\|f\|_\infty = \sup_{a \leq t \leq b} |f(t)|$.

Note that if f belongs to $C^{\alpha+\varepsilon}([a, b])$, with $\varepsilon > 0$, then (2.3) is true.

Let $g, f \in L^p([0, T])$ be two functions and $g_r^{b-} = g_r - g_{b-}$. In this case we say that f is integrable with respect to g if and only if $D_{a+}^\alpha f$ and $D_{b-}^{1-\alpha} g^{b-}$ exist, and

$(D_{a+}^\alpha f) D_{b-}^{1-\alpha} g^{b-} \in L^1([a, b])$. In this case we define the integral $\int_a^b f dg$ in the following way

$$\int_a^b f dg := \int_a^b (D_{a+}^\alpha f)(r) D_{b-}^{1-\alpha} g^{b-}(r) dr. \quad (2.4)$$

It is well-known that if $f \in C^\mu([a, b])$ and $g \in C^\beta([a, b])$, with $\mu + \beta > 1$, then it can be checked that $\int_a^b f_r dg_r$ is well-defined, and that it coincides with the Young's integral defined as a limit of Riemann sums.

3. The Stochastic Differential Equation

In this section we study the solution of the stochastic differential equation driven by a fBm with a discontinuous diffusion coefficient (1.1) as an approximation of a SDE. The integral in (1.1) is defined pathwise as a Young integral.

3.1. Initial condition. We divide our problem according to the initial condition x_0 in the following three cases:

- i) $x_0 > 0$,
- ii) $x_0 < 0$,
- iii) $x_0 = 0$.

Note that if equation (1.1) has a continuous solution x , then it satisfies the following:

Case i) The continuity of x implies that there exists $t_0 > 0$ such that $x_t > 0$ on $(0, t_0)$. Hence,

$$x_t = x_0 + \int_0^t \frac{1}{\alpha} dB_s^H = x_0 + \frac{1}{\alpha} B_t^H, \quad t \leq t_0.$$

Let \tilde{t}_0 be the first instant such that $x_{\tilde{t}_0} = 0$. Then,

$$x_t = x_0 + \frac{1}{\alpha} B_{\tilde{t}_0}^H + \int_{\tilde{t}_0}^t \sigma(x_s) dB_s^H = \int_{\tilde{t}_0}^t \sigma(x_s) dB_s^H, \quad t \geq \tilde{t}_0. \quad (3.1)$$

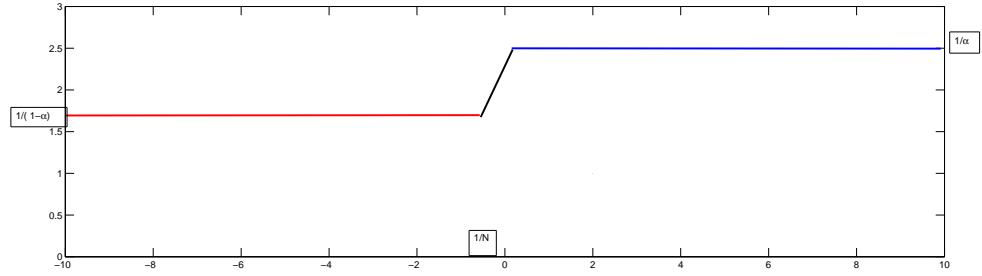
Case ii) Similarly, we have that, there is $t_1 > 0$ such that $x_t < 0$ on $(0, t_1)$. Consequently,

$$x_t = x_0 + \int_0^t \sigma(x_s) dB_s^H = x_0 + \frac{1}{1-\alpha} B_t^H, \quad t \leq t_1.$$

Again, let \tilde{t}_1 be the first time such that $x_{\tilde{t}_1} = 0$. Then,

$$x_t = \int_{\tilde{t}_1}^t \sigma(x_s) dB_s^H, \quad t \geq \tilde{t}_1. \quad (3.2)$$

So, we only need to consider the existence of a unique continuous solution to equation (3.2), for any $\tilde{t}_1 \geq 0$.

FIGURE 2. $\sigma^n(x)$ for $\alpha = 0.4$.

3.2. Notation and auxiliary results. In this section, we introduce the sequence of continuous functions $\{\sigma_n : \mathbb{R} \rightarrow \mathbb{R}_+ : n \in \mathbb{N}\}$ converging to σ given by:

$$\sigma_n(x) := \begin{cases} \sigma(x) & \text{if } x \notin (-1/n, 0) \\ \frac{1}{\alpha} + n \frac{1-2\alpha}{\alpha(1-\alpha)} x & \text{otherwise.} \end{cases}$$

Figure 2 shows the approximation function.

First, note that, σ_n is a Lipschitz continuous function with constant $n \frac{1-2\alpha}{\alpha(1-\alpha)}$, it means,

$$|\sigma_n(x) - \sigma_n(y)| \leq n \frac{1-2\alpha}{\alpha(1-\alpha)} |x - y|, \quad x, y \in \mathbb{R}. \quad (3.3)$$

Now we define, for $a \in \mathbb{R}$ fixed,

$$\Lambda_n(x) = \int_a^x \frac{ds}{\sigma_n(s)}, \quad x \in \mathbb{R}. \quad (3.4)$$

This integral is well defined due to $\sigma_n > 0$. Moreover, $\Lambda_n \in C^1(\mathbb{R})$ and $\Lambda'_n(x) = \frac{1}{\sigma_n(x)} > 0$, for all $x \in \mathbb{R}$. Therefore Λ_n is a strictly increasing function and uniformly bounded in compacts. As a consequence, Λ_n has an inverse Λ_n^{-1} .

Set

$$\Lambda(x) = \int_a^x \frac{ds}{\sigma(s)}, \quad x \in \mathbb{R}. \quad (3.5)$$

Remark 1. Since Λ is a strictly increasing function, then Λ^{-1} exists.

We will need the following auxiliary result later on.

Lemma 3.1. *There exists a positive constant C_α such that, for $x \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$|\Lambda_n(x) - \Lambda(x)| \leq \frac{C_\alpha}{n} \quad \text{and} \quad |\Lambda_n^{-1}(x) - \Lambda^{-1}(x)| \leq \frac{C_\alpha}{n}.$$

Proof. The definitions of σ and σ_n , together with (3.4) and (3.5), imply that

$$\Lambda(x) = \alpha(x - a)1_{\{x \geq 0\}} + ((1 - \alpha)x - a\alpha)1_{\{x < 0\}}$$

and

$$\begin{aligned}\Lambda_n(x) &= \alpha(x-a)1_{\{x \geq 0\}} \\ &+ \left(-a\alpha - \frac{\alpha(1-\alpha)}{n(1-2\alpha)} \left(\log\left(\frac{1}{\alpha}\right) - \log\left(\frac{1}{\alpha} + n\frac{1-2\alpha}{\alpha(1-\alpha)}\left(\frac{-1}{n} \vee x\right)\right) \right) \right. \\ &\left. + (1-\alpha)\left(\left(\frac{-1}{n} \wedge x\right) + \frac{1}{n}\right) 1_{\{x < 0\}}.\right.\end{aligned}$$

Consequently,

$$\Lambda^{-1}(x) = \left(\frac{x}{\alpha} + a\right) 1_{\{x \geq -a\alpha\}} + \left(\frac{x+a\alpha}{1-\alpha}\right) 1_{\{x \leq -a\alpha\}} \quad (3.6)$$

and

$$\begin{aligned}\Lambda_n^{-1}(x) &= \left(\frac{x}{\alpha} + a\right) 1_{\{x \geq -a\alpha\}} \\ &+ \frac{1}{\alpha} \left(\exp\left(\frac{n(1-2\alpha)(x+a\alpha)}{\alpha(1-\alpha)}\right) - 1 \right) \frac{\alpha(1-\alpha)}{n(1-2\alpha)} 1_{\{\alpha_n < x < -a\alpha\}} \\ &+ \left(\frac{x-\alpha_n}{1-\alpha} - \frac{1}{n}\right) 1_{\{x \leq \alpha_n\}},\end{aligned}$$

with

$$\alpha_n = -a\alpha - \frac{\alpha(1-\alpha)}{n(1-2\alpha)} \left(\log\left(\frac{1}{\alpha}\right) - \log\left(\frac{1}{1-\alpha}\right) \right).$$

Now, it is easy to finish the proof. □

Remark 2. Λ^{-1} is a Lipschitz function on \mathbb{R} . Indeed,

$$\begin{aligned}|\Lambda_n^{-1}(x) - \Lambda_n^{-1}(y)| &\leq \sup_{z \in K, n \in \mathbb{N}} |(\Lambda_n^{-1}(z))'| |x - y| \\ &= \left(\sup_{z \in K, n \in \mathbb{N}} |\sigma_n(\Lambda_n^{-1}(z))| \right) |x - y| \\ &\leq \left(\frac{1}{\alpha} + \frac{1}{1-\alpha} \right) |x - y|. \quad (3.7)\end{aligned}$$

Thus the claim is a consequence of Lemma 3.1.

We want to see that $x_t = \Lambda^{-1}(B_t^H - B_a^H + \bar{z})$, where $\bar{z} = \Lambda(0)$, is a solution of the equation

$$x_t = \int_a^t \sigma(x_s) dB_s^H, \quad t \geq a. \quad (3.8)$$

In order to study the uniqueness of the solution to (3.8), we consider the following auxiliary result.

Lemma 3.2. *Let $\tilde{\alpha} > 1 - H$ and $\gamma < H$ such that $\tilde{\alpha} > 1 - \gamma$. Then,*

$$\begin{aligned} |D_{t-}^{1-\tilde{\alpha}}(B^H)_s^{t-}| &\leq C_{\tilde{\alpha}} \|B^H\|_{\gamma,[0,T]} (t-s)^{\tilde{\alpha}+\gamma-1} \\ &\leq C_{\tilde{\alpha}} \|B^H\|_{\gamma,[0,T]} T^{\tilde{\alpha}+\gamma-1}, \quad 0 \leq s \leq t \leq T, \end{aligned}$$

where $(B^H)_s^{t-} = B_s^H - B_t^H$.

Proof. By definition of $D_{t-}^{1-\alpha}$ (see equalities (2.1) and (2.3)), we have

$$\begin{aligned} \left| D_{t-}^{1-\tilde{\alpha}}(B^H)_s^{t-} \right| &\leq \frac{1}{\Gamma(\tilde{\alpha})} \left(\frac{|B_s^H - B_t^H|}{(t-s)^{1-\tilde{\alpha}}} + (1-\tilde{\alpha}) \int_s^t \frac{|B_s^H - B_r^H|}{(r-s)^{2-\tilde{\alpha}}} dr \right) \\ &\leq C_{\tilde{\alpha}} \left(\|B^H\|_{\gamma,[0,T]} |t-s|^{\tilde{\alpha}+\gamma-1} + \|B^H\|_{\gamma,[0,T]} \int_s^t (r-s)^{\tilde{\alpha}+\gamma-2} dr \right) \\ &\leq C_{\tilde{\alpha}} \|B^H\|_{\gamma,[0,T]} |t-s|^{\tilde{\alpha}+\gamma-1}. \end{aligned}$$

Consequently, the proof is complete. \square

3.3. Existence and uniqueness for equation (1.1). In this section we state the main results of this article. The first one is related to the existence of a solution of the SDE (1.1), and the second one with the uniqueness of the solution.

Theorem 3.3. Existence: *Let $\gamma \in (\frac{1}{2}, H)$. Then there exists a pathwise solution $x \in C^\gamma([0, T])$ to the equation*

$$x_t = x_0 + \int_0^t \sigma(x_s) dB_s^H, \quad t \in [0, T]. \quad (3.9)$$

Proof. By (3.1) and (3.2), we only need to show that the equation

$$x_t = \int_a^t \sigma(x_s) dB_s^H, \quad t \geq a,$$

has a solution for every $a \geq 0$. Towards this end, we observe that, for $\alpha < 1/2$, (3.6) implies that Λ^{-1} is a convex function. Then, Remark 3.5 in [1] (see also Theorem 2.1 in [5]) yields

$$\Lambda^{-1}(B_t^H - B_a^H + \bar{z}) - \Lambda^{-1}(\bar{z}) = \int_a^t (\Lambda^{-1})'_+(B_s^H - B_a^H + \bar{z}) dB_s^H, \quad t \geq a.$$

Since $(\Lambda^{-1})'_+(x) = \sigma(x - \bar{z})$, setting $x_t = \Lambda^{-1}(B_t^H - B_a^H + \bar{z})$, then we have

$$x_t = \int_a^t \sigma(B_s^H - B_a^H) dB_s^H.$$

Finally, we also have that $\sigma(x) = \sigma(\Lambda^{-1}(x + \bar{z}))$. Thus the proof is complete. \square

Now we want to see that (3.9) has a unique solution $x \in C^\gamma([0, T])$, for $\gamma \in (\frac{1}{2}, H)$. Towards this end, in the following result we will use the approximation σ_n , because we will take advantage of [9] (Theorem 4.2.1), which requires the Lipschitz property of σ_n .

Note that we have figure out a solution of (3.9) such that

$$1_{\{a \leq r < s < T\}} \frac{\sigma(x_s) - \sigma(x_r)}{(s-r)^{1+\tilde{\alpha}}} \in L^1([0, T]^2), \quad (3.10)$$

with probability 1.

Lemma 3.4. *Let $\gamma \in (\frac{1}{2}, H)$ and x a γ -Hölder continuous solution to (3.9) such that (3.10) holds. Then,*

$$\Lambda_n(x_t) = \Lambda_n(x_0) + \int_0^t \frac{\sigma(x_s)}{\sigma_n(x_s)} dB_s^H, \quad t \in [0, T].$$

Proof. By [9] (Theorem 4.2.1) and (3.3), we obtain

$$\int_0^t \frac{dx_s}{\sigma_n(x_s)} = \lim_{|\pi| \rightarrow 0} \sum_{j=0}^{m-1} \frac{x_{s_{i+1}} - x_{s_i}}{\sigma_n(x_{s_i})},$$

where π is a partition of $[0, t]$ of the form $\pi = \{0 = s_0 < s_1 < \dots < s_m = t\}$.

Therefore, with the convention $\sigma_n^\pi(s) = \sum_{j=1}^{m-1} \frac{1_{[s_i, s_{i+1}]}(s)}{\sigma_n(x_{s_i})}$, we get

$$\begin{aligned} \int_0^t \frac{dx_s}{\sigma_n(x_s)} &= \lim_{|\pi| \rightarrow 0} \sum_{j=0}^{m-1} \int_{s_i}^{s_{i+1}} \frac{\sigma(x_r)}{\sigma_n(x_{s_i})} dB_r^H \\ &= \lim_{|\pi| \rightarrow 0} \int_0^t \sigma(x_r) \sigma_n^\pi(r) dB_r^H. \end{aligned} \quad (3.11)$$

On the other hand, the definition of Young integral (2.4) allows to establish

$$\begin{aligned} &\left| \int_0^t \left(\frac{\sigma(x_s)}{\sigma_n(x_s)} - \sigma_n^\pi(s) \sigma(x_s) \right) dB_s^H \right| \\ &\leq C_{\tilde{\alpha}} \left| \int_0^t \sigma(x_s) \left(D_{0+}^{\tilde{\alpha}} \left(\frac{1}{\sigma_n(x_s)} - \sigma_n^\pi(s) \right) \right) (D_{t-}^{1-\tilde{\alpha}} B^H)_s^{t-} ds \right| \\ &\quad + \tilde{\alpha} \left| \int_0^t \int_0^s \left(\frac{1}{\sigma_n(x_r)} - \sigma_n^\pi(r) \right) \frac{\sigma(x_s) - \sigma(x_r)}{(s-r)^{1+\tilde{\alpha}}} dr (D_{t-}^{1-\tilde{\alpha}} B^H)_s^{t-} ds \right| \\ &\leq C_{\tilde{\alpha}, T} \|B^H\|_{\gamma, [0, T]} \left(\int_0^T \left| D_{0+}^{\tilde{\alpha}} \left(\frac{1}{\sigma_n(x_s)} - \sigma_n^\pi(s) \right) \right| ds \right. \\ &\quad \left. + \int_0^T \int_0^s \left| \frac{1}{\sigma_n(x_r)} - \sigma_n^\pi(r) \right| \frac{|\sigma(x_s) - \sigma(x_r)|}{(s-r)^{1+\tilde{\alpha}}} dr ds \right), \end{aligned}$$

where last inequality follows from Lemma 3.2. Then, the result is a consequence of Zähle [9] (Theorems 4.1.1 and 4.3.1). \square

Note that the solution x to equation (3.9) is such that, there exists a random variable G such that

$$|x_s - x_t| \leq G|s - t|^\gamma.$$

In the following result we set

$$f_\gamma(s) = \mathbb{E}(G^{\frac{\tilde{\alpha}+\varepsilon}{\gamma}} |x_s|^{-\frac{(\tilde{\alpha}+\varepsilon)}{\gamma}}).$$

Lemma 3.5. *Let $\gamma \in (\frac{1}{2}, H)$, $1 - H < 1 - \gamma < \tilde{\alpha} < \gamma$, x a γ -Hölder continuous solution of (3.9) such that (3.10) and $\int_0^T 1_{\{x_s=0\}} ds = 0$ hold, $f_\gamma \in L^1([0, T])$ for some $\varepsilon > 0$ small enough, $G^{1-\eta} \in L^q$ and $\mathbb{P}(a < x_s \leq b) \leq g(s)(b - a)$, where $g \in L^{\frac{1}{p}}([0, T])$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $p \in (1, \frac{1}{1-\eta})$ for some $\eta < 1 - \frac{\tilde{\alpha}}{\gamma}$. Then,*

$$\Lambda(x_t) - \Lambda(x_0) = B_t^H, \quad t \in [0, T].$$

Remark 3. Note that the solution to (3.9) given in Theorem 3.3 satisfies the assumptions of this result.

Proof. From Lemmas 3.2, (3.1) and 3.4, we only need to prove that:

$$\int_0^T s^{-\tilde{\alpha}} \left| \frac{\sigma(x_s)}{\sigma_n(x_s)} - 1 \right| ds \longrightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (3.12)$$

and

$$I_{n_k} = \int_0^T \int_0^s \left| \frac{\sigma(x_s)}{\sigma_{n_k}(x_s)} - \frac{\sigma(x_r)}{\sigma_{n_k}(x_r)} \right| (s - r)^{-(1+\tilde{\alpha})} dr ds \rightarrow 0, \quad \text{as } n_k \rightarrow \infty, \quad (3.13)$$

for some subsequence $\{n_k : k \in \mathbb{N}\}$, w.p. 1. For (3.12), we have:

$$\int_0^T s^{-\tilde{\alpha}} \left| \frac{\sigma(x_s) - \sigma_n(x_s)}{\sigma_n(x_s)} \right| ds \leq C_\alpha \int_0^T s^{-\tilde{\alpha}} |\sigma(x_s) - \sigma_n(x_s)| ds.$$

Thus the dominated convergence theorem implies that (3.12) holds.

For (3.13), we have that

$$\left| \frac{\sigma(x_s)}{\sigma_n(x_s)} - \frac{\sigma(x_r)}{\sigma_n(x_r)} \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty, \text{ w.p. 1,}$$

and

$$\begin{aligned} \left| \frac{\sigma(x_s)}{\sigma_n(x_s)} - \frac{\sigma(x_r)}{\sigma_n(x_r)} \right| &= \left| \frac{\sigma(x_s)\sigma_n(x_r) - \sigma(x_r)\sigma_n(x_s)}{\sigma_n(x_s)\sigma_n(x_r)} \right| \\ &\leq C_\alpha |\sigma(x_s)\sigma_n(x_r) - \sigma(x_r)\sigma_n(x_s)| \\ &\leq C_\alpha |\sigma(x_s)| |\sigma_n(x_r) - \sigma_n(x_s)| + C_\alpha |\sigma_n(x_s)| |\sigma(x_s) - \sigma(x_r)| \\ &\leq C_\alpha (|\sigma_n(x_r) - \sigma_n(x_s)| + |\sigma(x_s) - \sigma(x_r)|) \\ &= I_1(s, r) + I_2(s, r). \end{aligned} \quad (3.14)$$

Let us start analyzing the term $I_2(s, r)$. In this case we have the following:

$$\begin{aligned} & I_{\{0 \leq r \leq s \leq T\}} I_2(s, r) (s - r)^{-(1+\tilde{\alpha})} \\ & \leq C_\alpha I_{\{0 \leq r \leq s \leq T\}} |r - s|^{-1-\tilde{\alpha}} \left(1_{\{x_s < 0 < x_r\}} + 1_{\{x_r < 0 < x_s\}} \right). \end{aligned} \quad (3.15)$$

Since over $\{x_s < 0 < x_r\}$, $|x_s| < |x_r - x_s|$ and $|x_r - x_s| \leq G|r - s|^\gamma$. Also we have $|r - s|^{-1} \leq G^{\frac{1}{\gamma}} |x_s|^{-\frac{1}{\gamma}}$. This inequality leads to:

$$I_{\{0 \leq r \leq s \leq T\}} I_2(s, r) (s - r)^{-(1+\tilde{\alpha})} I_{\{x_s < 0 < x_r\}} \leq C_\alpha |r - s|^{-1+\varepsilon} G^{\frac{\tilde{\alpha}+\varepsilon}{\gamma}} |x_s|^{-\frac{\tilde{\alpha}+\varepsilon}{\gamma}}. \quad (3.16)$$

Similarly we can obtain,

$$I_{\{0 \leq r \leq s \leq T\}} I_2(s, r) (s - r)^{-(1+\tilde{\alpha})} I_{\{x_r < 0 < x_s\}} \leq C_\alpha |r - s|^{-1+\varepsilon} G^{\frac{\tilde{\alpha}+\varepsilon}{\gamma}} |x_r|^{-\frac{\tilde{\alpha}+\varepsilon}{\gamma}}. \quad (3.17)$$

Now, we show that

$$I_1(s, r) \leq C_\alpha n^{1-\eta} G^{1-\eta} (s - r)^{\gamma(1-\eta)} \quad (3.18)$$

holds. By inequality (3.3), we can establish

$$\begin{aligned} & |\sigma_n(x_s) - \sigma_n(x_r)| \\ & = |\sigma_n(x_s) - \sigma_n(x_r)|^{1-\eta} |\sigma_n(x_s) - \sigma_n(x_r)|^\eta \\ & = n^{1-\eta} \left| \frac{1-2\alpha}{(1-\alpha)\alpha} \right|^{1-\eta} |x_s - x_r|^{1-\eta} |\sigma_n(x_s) - \sigma_n(x_r)|^\eta \\ & \leq C_{\alpha,\eta} n^{1-\eta} |x_s - x_r|^{1-\eta}, \end{aligned}$$

which allows us to see that (3.18) is true.

On the other hand,

$$\begin{aligned} & \mathbb{E} \left(I_{\{x_r < -\frac{1}{n} < x_s < 0\}} G^{1-\eta} \right) \\ & \leq \left(\mathbb{E} G^{q(1-\eta)} \right)^{1/q} \left(\mathbb{P} \left(x_s \in \left(-\frac{1}{n}, 0 \right) \right) \right)^{1/p} \leq C n^{-\frac{1}{p}} g_s^{\frac{1}{p}}. \end{aligned}$$

Hence, (3.18) implies

$$\mathbb{E} \left(\int_0^T \int_0^s I_1(s, r) (s - r)^{-(\tilde{\alpha}+1)} I_{\{x_r < -\frac{1}{n} < x_s < 0\}} dr ds \right) \rightarrow 0,$$

as $n \rightarrow \infty$. Moreover, this analysis also implies

$$\mathbb{E} \left(\int_0^T \int_0^s I_1(s, r) (s - r)^{-(\tilde{\alpha}+1)} \left(I_{\{x_s < -\frac{1}{n} < x_r < 0\}} + I_{\{-\frac{1}{n} < x_r, x_s < 0\}} \right) dr ds \right) \rightarrow 0,$$

as $n \rightarrow \infty$. Finally, we can see that inequalities (3.16) and (3.17) are also true when we write $I_1(s, r)$ instead of $I_2(s, r)$. Therefore, the result follows from (3.14)-(3.17) and the dominated convergence theorem. \square

4. Numerical Results

By (3.3) and the mean value theorem we have that $\sigma_n(\Lambda_n^{-1}(x))$ to be a Lipschitz function, then Zähle ([9], theorem 4.3.1) implies:

$$\begin{aligned} & \Lambda_n^{-1}(B_t^H - B_a^H + z_n) - \Lambda_n^{-1}(z_n) \\ &= \Lambda_n^{-1}(B_t^H - B_a^H + z_n) \\ &= \int_a^t \sigma_n(\Lambda_n^{-1}(B_s^H - B_a^H + z_n)) dB_s^H, \quad t \geq a, \end{aligned} \quad (4.1)$$

with $z_n = \Lambda_n(0)$.

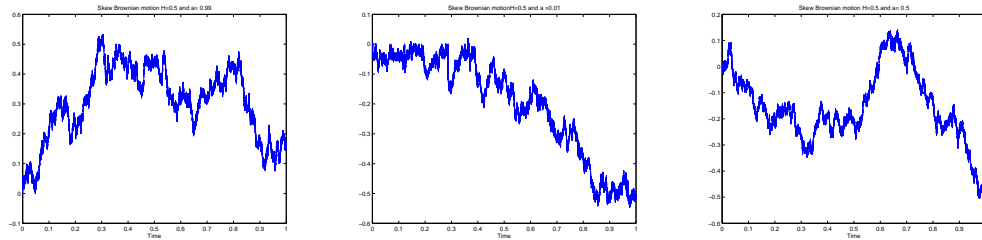
Therefore, $x_t^{(n)} = \Lambda_n^{-1}(B_t^H - B_a^H + z_n)$ is a solution to the equation

$$x_t^{(n)} = \int_a^t \sigma_n(x_s^{(n)}) dB_s^H, \quad t \geq a. \quad (4.2)$$

Consequently, by Lemma 3.1, we can approximate the solution of equation (1.1) by the solution of equation (4.2).

In this section we show some simulations for the unique solution of equation (1.1) for different values of H and α . We use the approximation given in (4.2). We also present the limit case $H = 0.5$, and $a = 0$.

The following figures correspond to the particular case the Skew Brownian motion (SBm) as the solution of a SDE with discontinuous coefficient in the diffusion. Also, the case (c), $\alpha = 0.5$ corresponds to the Brownian motion process.

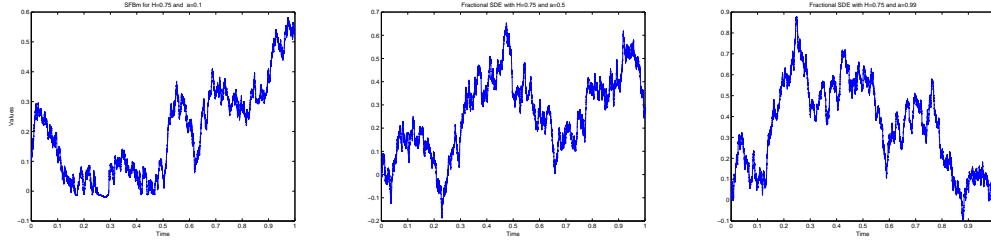


(a) SBm for $\alpha = 0.99$.

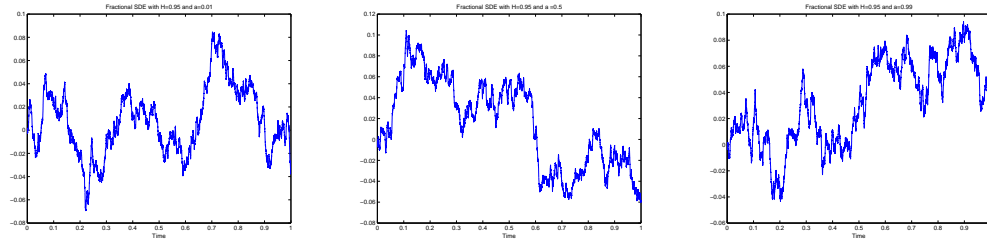
(b) SBm for $\alpha = 0.01$.

(c) SBm for $\alpha = 0.5$.

The following pictures shows samples for the unique solution of (1.1), for different values of H and α .



(d) x_t for $H = 0.75$ and $\alpha = 0.1$ (e) x_t for $H = 0.75$ and $\alpha = 0.5$ (f) x_t for $H = 0.75$ and $\alpha = 0.99$



(g) x_t for $H = 0.95$ and $\alpha = 0.1$ (h) x_t for $H = 0.95$ and $\alpha = 0.5$ (i) x_t for $H = 0.95$ and $\alpha = 0.99$

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